TOPOLOGY QUALIFYING EXAM – AUGUST 2021

Instructions: Solve any FOUR out of the five problems. If you work on all five problems, your top four problem scores will count towards your exam score.

NOTE: No books, notes, or resources of any kind may be used during the exam.

All work must be shown, and all answers justified appropriately.

1. Suppose that f is a continuous map from a sphere S^n to the torus $T^2 = S^1 \times S^1$.
(a) If $n \geq 2$, prove that the map f is nullhomotopic (i.e. homotopic to a constant map).

(b) If $n = 1$, does f have to be nullhomotopic? Justify your answer.

2. (a) Explain how to construct a CW complex X with fundamental group $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

(b) Compute all the homology groups of the CW complex X you constructed in part (a).

(c) Is the answer to part (b) the same for all CW complexes Y with fundamental group $\pi_1(Y) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$?

3. Prove or find a counterexample for each of the following assertions:

(a) Suppose that for connected CW-complexes X, Y, we have that $H_i(X) = H_i(Y)$ for all $i \geq 0$. Then X and Y are homotopy equivalent.

(b) Let $Z = \mathbb{R}P^2 \vee \mathbb{R}P^2$ be the wedge of two projective planes. There is a covering space W of Z such that $\pi_1(W) = \mathbb{Z}$.

(c) There is no retraction $r : D^2 \to S^1$ from the disk to its boundary circle.

4. Prove that if $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$, then $m = n$.

5. Recall that the suspension SX of a space X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Prove that $\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X)$ for all $n \geq 0$.